Math 102

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Announcements

- Midterm - we're still working on it!
  - Technical hitch, hopefully it is resolved quickly - expect midterms later this week.
  - The marks and feedback will come back to you electronically.

Last Time

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$$

Goals Today

- What is an inverse function?
- Logarithms
  - As the inverse of an exponential function
  - Applications
- More examples of exponential growth and decay
  - Population growth
  - Radioactive decay
  - etc.

Inverse Functions - definition

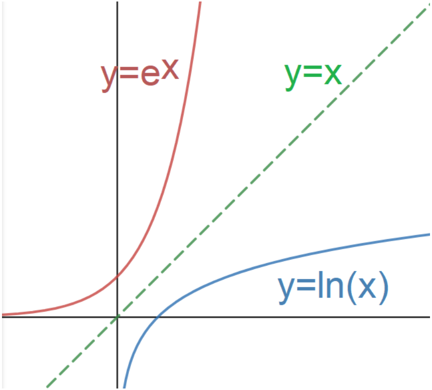
$f(x)$	$f^{-1}(x)$
$x+3$	$x-3$
$5x$	$\frac{x}{5}$
x^3	$x^{1/3} = \sqrt[3]{x}$
x^3+3	$(x-3)^{1/3}$
e^x	$\ln(x)$

The inverse of a function is another function which 'undoes' the first function.

$$f^{-1}(f(x)) = x$$

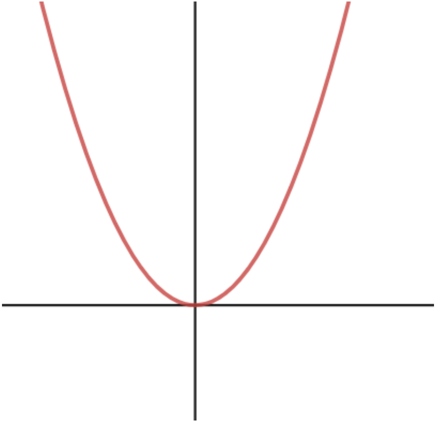
To find the inverse: write down $y = f(x)$ and try to solve for x. If $y = f(x)$, then $f^{-1}(y) = x$.

Inverse Functions - graphically

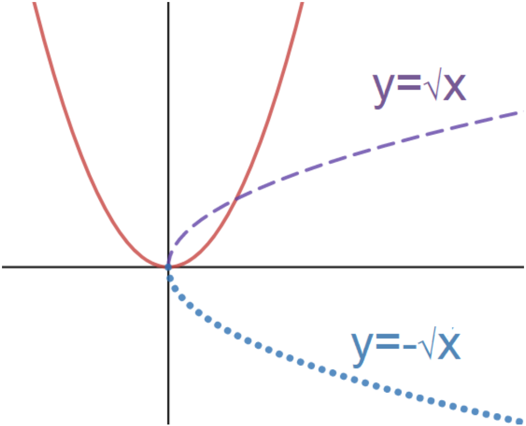


$$y = f(x) \implies x = f^{-1}(y)$$

Therefore, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another across the line $y = x$.

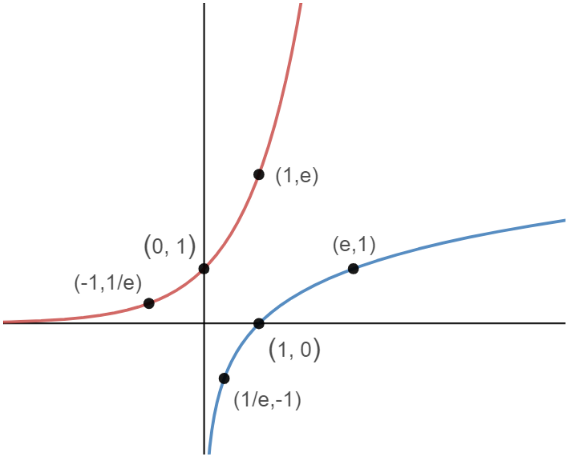


Question: Does the function $y = x^2$ have an inverse?



Question: Does the function $y = x^2$ have an inverse?

Answer: It does not! In order for a function $f(x)$ to have an inverse, the reflection of its graph must pass the vertical line test. Therefore, $y = f(x)$ must pass the horizontal line test.



e^x has domain $(-\infty, \infty)$ and range $(0, \infty)$.
$\ln(x)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.

Rules of exponentials and logarithms

e^x	$\ln(x)$
$e^a \cdot e^b = e^{a+b}$	$\ln(A \cdot B) = \ln(A) + \ln(B)$
$(e^a)^b = e^{ab}$	$\ln(A^b) = b\ln(A)$

You can deduce the rules for $\ln(x)$ using the rules for e^x if you let $A = e^a$ and $B = e^b$.

Question: A bacterial colony grows at a rate proportional to the size (exponential growth).

- At time $t = 0$, there are 100 million cells.
- After 3 hours ($t = 3$), there are 500 million cells.

How many cells are there at $t = 7$?

Question: Let $B(t)$ be an exponential function.

- $B(0) = 100$.
- $B(3) = 500$.

What is $B(7)$?

Question: Let $B(t)$ be an exponential function.

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What is $B(7)$?

Solution 1: Let $B(t) = Ca^t$.

$$B(0) = 100 \implies C = 100$$

$$B(3) = 500 \implies Ca^3 = 500 \implies a = 5^{1/3}$$

Therefore, $B(t) = 100 \cdot 5^{t/3}$. So $\boxed{B(7) = 100 \cdot 5^{7/3}}$.

Question: Let $B(t)$ be an exponential function.

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What is $B(7)$?

Solution 2: Let $B(t) = Ce^{kt}$.

$$B(0) = 100 \implies C = 100$$

$$B(3) = 500 \implies Ce^{3k} = 500 \implies k = \frac{\ln(5)}{3}$$

Therefore, $B(t) = 100 \cdot e^{\frac{t\ln(5)}{3}}$. So $\boxed{B(7) = 100 \cdot e^{\frac{7\ln(5)}{3}}}$.

Question: Consider the same bacterial colony

$$B(t) = 100 \cdot 5^{t/3} = 100 \cdot e^{\frac{t \ln(5)}{3}}$$

For what value of t is $B(t) = 200$?

Logarithms in other bases

$\log_a(b)$: 'to what number must I raise a to get b?'
For example,

$$\log_2(8) = 3 \qquad \log_3(9) = 2 \qquad \log_5(5) = 1$$

$$\log_8(2) = \frac{1}{3} \qquad \log_3(1/9) = -2 \qquad \log_4(1) = 0$$

$$\ln(b) = \ln(a^{\log_a(b)}) = \ln(a) \cdot \log_a(b)$$

$$\implies \log_a(b) = \frac{\ln(b)}{\ln(a)}$$

Question: Consider the same bacterial colony

$$B(t) = 100 \cdot 5^{t/3} = 100 \cdot e^{\frac{t \ln(5)}{3}}$$

For what value of t is $B(t) = 200$?

Solution 1: Let t be such that $B(t) = 200$. Then

$$100 \cdot e^{\frac{t \ln(5)}{3}} = 200$$

$$e^{\frac{t \ln(5)}{3}} = 2$$

$$\frac{t \ln(5)}{3} = \ln(2)$$

$$\boxed{t = \frac{3 \ln(2)}{\ln(5)}}$$

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Solution 1: Let t be such that $B(t) = 200$. Then

$$100 \cdot 5^{t/3} = 200$$

$$5^{t/3} = 2$$

$$t/3 = \log_5(2)$$

$$\boxed{t = 3\log_5(2)}$$

Question: A radioactive element decays at a rate proportional to the quantity present, denoted $R(t)$. Suppose that $R(1) = 5$ and $R(2) = 3$. Calculate the **half-life**, i.e. the amount of time required for the element to decrease by half.

Question: A radioactive element decays at a rate proportional to the quantity present, denoted $R(t)$. Suppose that $R(1) = 5$ and $R(2) = 3$. Calculate the **half-life**, i.e. the amount of time required for the element to decrease by half.

Answer: $\log_{3/5}(1/2) = \frac{\ln(1/2)}{\ln(3/5)} = \frac{-\ln(2)}{\ln(3)-\ln(5)}$. You can see this because if you let $R(t) = Ca^t$, then $R(1) = Ca$ and $R(2) = Ca^2$. So

$$\frac{R(2)}{R(1)} = \frac{Ca^2}{Ca} = a$$

So $a = 3/5$. The half-life is the number h such that $(3/5)^h = 1/2$, so $h = \log_{3/5}(1/2)$.

Question: A drug is administered by IV to a patient at a constant rate R. At the same time, the drug is metabolized in the body at a rate proportional to the concentration in the body.
Let $C(t)$ denote the quantity of drug in the patient's body at time t. As $t \to \infty$, what do you think will happen to $C(t)$?

- It will go to infinity.
- It will level off to some positive value.
- It will eventually go to zero.
- It will oscillate.

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Recap

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